Mathematics III
Part II (Honrs)

Concept of a Subgroup an cyclic group.

Subgroup:— Let (G,o) be a group and H is a non-empty subset of G. Then H is said to be subgroup of G iff H is a group itself under the group operation o of G.

i.e. the group (H,o) is a subgroup of the group (G,o) if $H \subseteq G$ and H is non empty.

i.e. The set C of complex numbers is a group under addition. The set R of Real numbers is a subset of C and is a group under addition. So R is a subgroup of C.

i.e. the set of real numbers is a subgroup of the additive group of complex numbers.

Cyclic group:— A group (G,o) is said to be a cyclic group if there exists an element $a \in G$ such that every element b of G can be expressed as an integral power a^n of a for some integer n. The set G is called a generator of cyclic group i.e. a is generator of G. It is denoted by

$G = \langle a \rangle$

i.e. (i) The set $G = \{1, \omega, \omega^2\}$ i.e. cube roots of unity is a cyclic group under multiplication of complex number.

This is because we can write

$G = \{\omega^3, \omega^1, \omega^2\}$

i.e. G is and integral power of $\omega \in G$.

Theorem:— Prove that a non-empty subset H of the group G under the operation 'o' forms a subgroup of G iff $a, b \in H \Rightarrow a o b^{-1} \in H$.

Proof:— Necessary condition:— let H be a subgroup of the group G. Then we have to prove that

$$a \in H, b \in H \Rightarrow ab^{-1} \in H$$

Given that H is a group and $b \in H$ then $b^{-1} \in H$.
We have $a \in H$, $b^{-1} \in H$ then by closure law $ab^{-1} \in H$.

So the condition is necessary.

Sufficient Condition:. Let $a \in H, b \in H \Rightarrow ab^{-1} \in H$
Then we have to prove that H is a subgroup of the group G.

(i) Existence of identity: If $a \in H, a \in H$
$\Rightarrow aa^{-1} \in H \Rightarrow e \in H$. Where e is identity element of G.
So identity element exists in H.

(ii) Existence of Inverse: $e \in H, a \in H \Rightarrow ea^{-1} \in H$
Thus the inverse axiom is satisfied by H.

(iii) closure law: $a \in H, b \in H \Rightarrow a \in H, b^{-1} \in H$
(as existence of inverse hold for H so $b^{-1} \in H$)
$\therefore a \in H, b \in H \Rightarrow a(b^{-1})^{-1} \in H$
$\Rightarrow ab \in H$ $\therefore$ closure law hold.

(iv) Associative law: Since G is a group then G ~~be~~ must be associative $H \subset G$ then H must also be associative. Hence H satisfies all the axioms of a group. But $H \subset G$.
So H is subgroup of a group G.

Proved.